Math 8	Instructor: Padraic Bartlett
Series	
Week 3	*Caltech 2012*

1 Series of Positive Numbers

1.1 Series: Definitions and Tools

Recall the last sequence we studied, as an example of when to use the Cauchy criterion: $a_n = \sum_{k=1}^{n} \frac{1}{k^2}$. As it turns out, we chose to end on this example for a very specific reason: not only was it the last sequence example we studied, it was the first example of a **series**! We define series as follows:

Definition. A sequence is called **summable** if the sequence $\{s_n\}_{n=1}^{\infty}$ of partial sums

$$s_n := a_1 + \ldots a_n = \sum_{k=1}^{n} a_k$$

converges.

If it does, we then call the limit of this sequence the **series** associated to $\{a_n\}_{n=1}^{\infty}$, and denote this quantity by writing

$$\sum_{n=1}^{\infty} a_n.$$

We say that a series$\sum_{n=1}^{\infty} a_n$ **converges** or **diverges** if the sequence $\{\sum_{k=1}^{n} a_k\}_{n=1}^{\infty}$ of partial sums converges or diverges, respectively.

Just like sequences, we have a collection of various tools we can use to study whether a given series converges or diverges. Here are two such tools:

1. **Comparison test**: If $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ are a pair of sequences such that $0 \leq a_n \leq b_n$, then the following statement is true:

$$\left(\sum_{n=1}^{\infty} b_n \text{ converges}\right) \Rightarrow \left(\sum_{n=1}^{\infty} a_n \text{ converges}\right).$$

When to use this test: when you're looking at something fairly complicated that either (1) you can bound above by something simple that converges, like $\sum 1/n^2$, or (2) that you can bound below by something simple that diverges, like $\sum 1/n$.

2. **Ratio test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = r,$$

then we have the following three possibilities:

- If $r < 1$, then the series $\sum_{n=1}^{\infty} a_n$ converges.
- If $r > 1$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.
- If $r = 1$, then we have no idea; it could either converge or diverge.

When to use this test: when you have something that is growing kind of like a geometric series: so when you have terms like 2^n or $n!$.

To illustrate how to work with these definitions, we work a collection of examples here:

1.2 Series: Example Calculations

Claim 1. *(Definition example 1:) The series*

$$\sum_{n=1}^{\infty} n$$

diverges.

Proof. For any n, the sum of the numbers from 1 to n is just $\frac{n(n+1)}{2}$; therefore, by definition, we know that

$$\sum_{n=1}^{\infty} n = \lim_{n \to \infty} \left(\sum_{k=1}^{n} k \right) = \lim_{n \to \infty} \frac{n(n+1)}{2} = \infty,$$

and therefore diverges. □

Claim 2. *(Definition example 2:) The* ***harmonic series***

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Proof. Notice the following inequalities:

$$\begin{aligned}
\left(\frac{1}{2}\right) &\geq \frac{1}{2} = \frac{1}{2}, \\
\left(\frac{1}{3} + \frac{1}{4}\right) &\geq 2 \cdot \frac{1}{4} = \frac{1}{2}, \\
\left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) &\geq 4 \cdot \frac{1}{8} = \frac{1}{2}, \\
\left(\frac{1}{9} + \ \ldots \ + \frac{1}{16}\right) &\geq 8 \cdot \frac{1}{16} = \frac{1}{2}, \\
&\vdots
\end{aligned}$$

In particular, notice that by applying these inequalities, we can show that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots \\
&\geq 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n}\right) \\
&\geq 1 + \frac{1}{2} + \dots + \frac{1}{2} \\
&= 1 + \frac{n}{2},
\end{aligned}$$

for **any** value of n. Therefore, we know that this series must diverge, because there is no possible limit value L that is both finite and greater than $1 + \frac{n}{2}$ for every value of n. □

Claim 3. *(Comparison test example): If $\{a_n\}_{n=1}^{\infty}$ is a sequence of positive numbers such that the series $\sum_{n=1}^{\infty} a_n$ converges, then the series*

$$\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$$

must also converge.

Proof. To see why, simply notice that each individual term $\frac{\sqrt{a_n}}{n}$ in this series is just the geometric mean of a_n and $\frac{1}{n^2}$. Because both of the series$\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are convergent, we would expect their average (for any reasonable sense of average) to also converge: in particular, we would expect to be able to bound its individual terms above by some combination of the original terms a_n and $\frac{1}{n^2}$!

In fact, we actually **can** do this, as illustrated below:

$$\begin{aligned}
& 0 \leq \left(\sqrt{a_n} - \frac{1}{n}\right)^2 \\
\Rightarrow \quad & 0 \leq a_n + \frac{1}{n^2} - 2\frac{\sqrt{a_n}}{n} \\
\Rightarrow \quad & \frac{\sqrt{a_n}}{n} \leq \frac{1}{2}\left(a_n + \frac{1}{n^2}\right) < a_n + \frac{1}{n^2}.
\end{aligned}$$

Look at the series $\sum_{n=1}^{\infty}\left(a_n + \frac{1}{n^2}\right)$. Because both of the series

$$\sum_{n=1}^{\infty} a_n, \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converge, we can write

$$\sum_{n=1}^{\infty}\left(a_n + \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and therefore notice that this series also converges; therefore, by the comparison test, our original series $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ must also converge. □

Claim 4. *(Ratio test example): The series*

$$\sum_{n=1}^{\infty} \frac{2^n \cdot n!}{n^{n+1}}$$

converges.

Proof. Motivated by the presence of both a $n!$ and a 2^n, we try the ratio test:

$$\begin{aligned}
\frac{a_n}{a_{n-1}} &= \frac{\frac{2^n \cdot n!}{n^{n+1}}}{\frac{2^{n-1}\cdot(n-1)!}{(n-1)^n}} \\
&= \frac{2^n \cdot n! \cdot (n-1)^n}{2^{n-1} \cdot (n-1)! \cdot n^{n+1}} \\
&= \frac{2 \cdot n \cdot (n-1)^n}{n^{n+1}} \\
&= \frac{2 \cdot (n-1)^n}{n^n} \\
&= 2 \cdot \left(\frac{n-1}{n}\right)^n \\
&= 2 \cdot \left(1 - \frac{1}{n}\right)^n
\end{aligned}$$

Here, we need one bit of knowledge that you may not have encountered before: the fact that the limit

$$\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x,$$

and in particular that

$$\lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^n = \frac{1}{e}.$$

(Historically, I'm pretty certain that that this is how e was defined; so feel free to take it as a definition of e itself.)

Applying this tells us that

$$\lim_{n\to\infty} \frac{a_n}{a_{n-1}} = \lim n \to \infty 2 \cdot \left(1 - \frac{1}{n}\right)^n = \frac{2}{e},$$

which is less than 1. So the ratio test tells us that this series converges! □

2 Series with Negative Terms

One somewhat-frustrating part of the two tests listed above is that both of them require the terms of the series they study to be positive. However, in practice we often find ourselves having to deal with series that contain negative terms along with positive terms. How can we study such series? The following two tools help with this:

1. **Alternating series test**: If $\{a_n\}_{n=1}^{\infty}$ is a sequence of numbers such that

   - $\lim_{n\to\infty} a_n = 0$ monotonically, and
   - the a_n's alternate in sign, then

   the series $\sum_{n=1}^{\infty} a_n$ converges.

   When to use this test: when you have an alternating series.

2. **Absolute convergence ⇒ convergence**: Suppose that $\{a_n\}_{n=1}^{\infty}$ is a sequence such that

   $$\sum_{n=1}^{\infty} |a_n|$$

   converges. Then the sequence $\sum_{n=1}^{\infty} a_n$ also converges.

   When to use this test: whenever you have a sequence that has positive and negative terms, that is not alternating. (Pretty much every other test requires that your sequence is positive, so you'll often apply this test and then apply one of the other tests to the series $\sum_{n=1}^{\infty} |a_n|$.)

We illustrate the use of these two tests here:

Claim 5. *(Alternating series test): The series*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

converges.

Proof. The terms in this series are alternating in sign: as well, they're bounded above and below by $\pm\frac{1}{n}$, both of which converge to 0. Therefore, we can apply the alternating series test to conclude that this series converges. □

Claim 6. *(Absolute convergence ⇒ convergence): The series*

$$\sum_{n=1}^{\infty} \frac{\cos^n(nx)}{n!}$$

converges.

Proof. We start by looking at the series composed of the absolute values of these terms:

$$\sum_{n=1}^{\infty} \frac{|\cos^n(nx)|}{n!}$$

Because $|\cos(x)| \leq 1$ for all x, we can use the comparison test to notice that this series will converge if the series

$$\sum_{n=1}^{\infty} \frac{1}{n!}$$

converges.

We can study this series with the ratio test:

$$\lim_{n\to\infty} \frac{\frac{1}{n!}}{\frac{1}{(n-1)!}} = \lim_{n\to\infty} \frac{1}{n} = 0,$$

which is less than 1. Therefore this series converges, and therefore (by the comparison test + absolute convergence $\Rightarrow$ convergence) our original series

$$\sum_{n=1}^{\infty} \frac{\cos^n(nx)}{n!}$$

converges. $\square$

2.1 Rearranging Sums

We make a quick detour here to some more philosopical questions. Think, for a moment about what series **are**: just infinite sums of things! So: with finite sums, we know that addition has several nice properties. One particularly nice property that addition has is that it's **commutative**: i.e. the order in which we add things up doesn't matter! In other words, we know that

$$1 + 2 + 3 = 3 + 2 + 1 = 6.$$

A natural question we could ask, then, is the following: does this hold true with series? In other words, if we rearrange the terms in the series (say)

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n},$$

will it still sum up to the same thing?

Well: let's try! Specifically, consider the following way to rearrange our series:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots \\ &=^{?} \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) \dots\end{aligned}$$

In the second rearrangement, we've ordered terms in the following groups:

$$\dots + \left(\frac{1}{\text{odd number}} - \frac{1}{2 \cdot \text{that odd number}} - \frac{1}{2 \cdot \text{that odd number} + 2}\right) + \dots$$

Notice that every term from our original series shows up in exactly one of these groups. Specifically, each odd number clearly shows up once: as well, for any even number, there are two cases: either it has exactly one factor of 2, in which case it's of the form (2$\cdot$an odd number) and shows up exactly once, or it's a multiple of 4, in which case it shows up as a (2$\cdot$an odd number+2), and also shows up once. So this is in fact a proper rearrangement! We haven't forgotten any terms, nor have we repeated any terms.

But, if we group terms as indicated below in our rearrangement, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} &=^{?} \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \left(\frac{1}{7} - \frac{1}{14} - \frac{1}{16}\right) \dots \\
&= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} \dots \\
&= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} \dots \\
&= \frac{1}{2} \cdot \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} \dots\right) \\
&= \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}.
\end{aligned}$$

So: if rearranging terms doesn't change the sum of an infinte series, we've just shown that

$$\sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n} = \frac{1}{2} \cdot \sum_{n=1}^{\infty} \frac{(1)^{n+1}}{n}.$$

The only number that is equal to half of itself is 0: therefore, this series must sum to 0!

However: look at our series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ again. In specific, if we just expand this sum without rearranging anything, we can see that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \dots \\
&= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{9} - \frac{1}{10}\right) + \dots \\
&= \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \frac{1}{5 \cdot 6} + \frac{1}{7 \cdot 8} + \frac{1}{9 \cdot 10} + \dots \\
&\geq \frac{1}{2}.
\end{aligned}$$

So this sum definitely **cannot** converge to 0, as all of its partial sums are $\geq \frac{1}{2}$! This answers our question earlier about rearranging series fairly definitively: we've just shown that rearranging series can do **unpredictable, terrible, and horrible** things to the series itself.

In fact, we have the following two theorems about what happens when series are rearranged:

Theorem. Suppose that $\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series. Then rearranging the a_n's does not change what our series converges to.

Theorem. Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series that is not absolutely convergent. Then for any $r \in \mathbb{R} \cup \{\pm\infty\}$, we can rearrange the a_n's so that the resulting series converges to r.

Cool!